Transcendence of an algebraic point of the several variable function

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Duverney and Nishioka [1] gave us strong criterion for transcendence for Mahler type function. If α is an algebraic number, we denote its house by $\overline{|\alpha|} = \max\{|\alpha^\sigma| \mid \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})\}$ and by $\mathrm{den}\,(\alpha)$ the least positive integer such that $\mathrm{den}\,(\alpha)\,\alpha$ is an algebraic integer, and we set $\|\alpha\| = \max\{\overline{|\alpha|}, \mathrm{den}\,(\alpha)\}$. Let $\boldsymbol{K}$ be an algebraic number field and $O_{\boldsymbol{K}}$ the ring of integers in $\boldsymbol{K}$. They considered the following function

$$\Phi_0(x) = \sum_{k\geq 0} \frac{E_k(x^{r^k})}{F_k(x^{r^k})},$$

where

$$E_k(x) = a_{k1}x + a_{k2}x^2 + \cdots + a_{kL}x^L \in \boldsymbol{K}[x],$$
$$F_k(x) = 1 + b_{k1}x + b_{k2}x^2 + \cdots + b_{kL}x^L \in O_{\boldsymbol{K}}[x],$$
$$\log\|a_{kl}\|, \log\|b_{kl}\| = o(r^k),\ 1 \leq l \leq L.$$

Then they proved the following:

Transcendence criterion (Duverney and Nishioka [1]). *Let α be an algebraic number with $0 < |\alpha| < 1$ such that $F_k(\alpha^{r^k}) \neq 0$ for every $k \geq 0$, then $\Phi_0(\alpha)$ is an algebraic number if and only if $\Phi_0(x)$ is a rational function.*

They showed necessary and sufficient condition of Mahler type function by introducing inductive method. The author [2] showed the transcendence criterion for several variable Mahler type function under some restricted conditions. Tachiya [4] gave transcendence criterion for infinite product case as same as Nishioka and Duverney. Moreover, he also gave transcendence criterion for the infinite product case for several variable function in [3]. The author will give complete generalisation of transcendence criterion of Nishioka and Duverney.

We use usual notations

$$|\boldsymbol{\lambda}| = \sum_{i=1}^{m} \lambda_i, \quad \boldsymbol{\alpha}^{\boldsymbol{\lambda}} = \prod_{i=1}^{m} {\alpha_i}^{\lambda_i}, \quad \text{and} \quad \langle \boldsymbol{\lambda}, \boldsymbol{\eta} \rangle = \sum_{i=1}^{m} \lambda_i \eta_i$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m), \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_m)$. Let $r \geq 2$ be a integer. We define $\Omega_n \boldsymbol{z} := ({z_1}^{r^n}, \ldots, {z_m}^{r^n})$ for $\boldsymbol{z} = (z_1, \ldots, z_m)$ and

$$S := \Phi_0(\boldsymbol{z}) = \sum_{k \geq 0} \frac{E_k(\Omega_k \boldsymbol{z})}{F_k(\Omega_k \boldsymbol{z})} \in \boldsymbol{K}[[\boldsymbol{z}]] = \boldsymbol{K}[[z_1, \ldots, z_m]], \tag{1}$$

where

$$E_k(\boldsymbol{z}) = \sum_{1 \leq |\boldsymbol{\lambda}| \leq L} a_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}}, \quad F_k(\boldsymbol{z}) = 1 + \sum_{1 \leq |\boldsymbol{\lambda}| \leq L} b_{k\boldsymbol{\lambda}} \boldsymbol{z}^{\boldsymbol{\lambda}} \in \boldsymbol{K}[\boldsymbol{z}]$$

with

$$\log \|a_{k\boldsymbol{\lambda}}\|, \log \|b_{k\boldsymbol{\lambda}}\| = o(r^k).$$

Theorem 1. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be an algebraic point with $0 < |\alpha_i| < 1$ $(1 \leq i \leq m)$ and $|\alpha_1|, \ldots, |\alpha_m|$ are multiplicatively independent. If $F_k(\Omega_k \boldsymbol{\alpha}) \neq 0$ for every $k \geq 0$, then $\Phi_0(\boldsymbol{\alpha})$ is an algebraic number if and only if $\Phi_0(\boldsymbol{z})$ is a rational function.*

References

[1] D. Duverney and Ku. Nishioka. An inductive method for proving the transcendence of certain series. *Acta. Arithmetica*, 110(4):305–330, 2003.

[2] Takeshi Kurosawa. Transcendence of certain series involving binary linear recurrences. *Journal of Number Theory*, 123(1):35–58, 2007.

[3] Yohei Tachiya. Transcendence of the values of infinite products in several variables. to appear.

[4] Yohei Tachiya. Transcendence of certain infinite products, 2007. to appear.